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MAY 1974 AIAA JOURNAL VOL. 12, NO. 5

Response of Separation to Impulsive Changes of Outer Flow

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The unsteady boundary-layer equations are solved numerically for an impulsive change of the outer inviscid flow. The vanishing of the skin friction is not related to separation in unsteady flows and for this purpose an upwind differencing scheme is employed to integrate the boundary-layer equations through regions of partially reversed flow. The point of separation, as defined by Sears and Telionis, and the point of zero skin friction are plotted vs time. It is found that the point of zero skin friction jumps all the way to the leading edge immediately after the impulsive change of the outer flow and travels downstream for subsequent times. The point of separation instead remains at its original location, and only after some time elapses, it starts moving upstream. Both points asymptotically tend to the location of steady-state separation that corresponds to the new distribution of the outer flow.

Nomenclature

A = outer flow velocity gradient, see Eq. (16)
F = normalized longitudinal velocity component, $F = (u/U_e)$ at time t
F^0 = normalized longitudinal velocity component, $F^0 = (u/U_e)$ at time $t - \Delta t$
L = typical length of the problem
n = nondimensional distance normal to body surface
N = stretched normal coordinate, $N = n(Re)^{1/2}$
Re = Reynolds number
$S_u(t)$ = physical distance of unsteady separation point measured along the body surface from leading edge
$S_{zf}(t)$ = physical distance of the point of zero skin friction measured along the body surface from leading edge
s = nondimensional distance measured along the body surface from leading edge
t = time nondimensionalized with $\tau = (L/U_\infty)$
u = nondimensional viscous flow velocity component in s-direction
U = inviscid flow velocity component in s-direction
v = nondimensional viscous velocity component in n-direction
V = modified viscous velocity component in n-direction
α_j' = coefficients of the difference momentum equation for unsteady state, see Eq. (31)
α_j = coefficients of the difference momentum equation in the equivalent steady-state form, see Eq. (33)
β = pressure gradient parameter
η = transformed stretched normal coordinate
ξ = transformed surface coordinate

Subscripts

e = outer flow conditions
∞ = freestream conditions
H = initial steady-state solution

Received June 28, 1973; presented as Paper 73-684 at the AIAA 6th Fluid and Plasma Dynamics Conference, Palm Springs, Calif., July 16–18, 1973; revision received December 13, 1973. Research supported by the Air Force Office of Scientific Research, under Grant AFOSR-72-2361. The U.S. Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation hereon.

Index categories: Boundary Layers and Convective Heat Transfer—Laminar; Nonsteady Aerodynamics.

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1. Introduction

THE importance of unsteady viscous flow problems has been widely recognized as reflected by the large number of recent publications on the topic.[1] Yet one of the characteristic features and a peculiar quirk of such flows, the phenomenon of separation, had not been properly defined more than 50 years after research in unsteady boundary layers was initiated. The location of actual separation regulates phenomena like the rotating stall in axial compressors or the stalling flutter of an air foil. This paper is definitely not the answer to all irksome questions on the topic. It provides though some encouraging indications that the research in this area is on the right path and supplies the results of calculations that could be easily reproduced experimentally for comparison.

For 60 years of research in boundary-layer theory, Prandtl's criterion[2] of vanishing skin friction has been used to predict effectively the phenomenon of separation. Only in the late fifties some concern was registered by Moore,[3] Rott[4] and Sears[5] about the validity of this criterion. It was then recognized that the classical criterion may lead to erroneous results in cases other than two-dimensional flow over fixed walls. Hartunian and Moore[6] have considered the case of a slowly moving point of separation. The similarity of the flow in the neighborhood of separation, for steady flow over fixed walls, was also pointed out. Vidal[7] and Ludwig[8] experimentally verified the theoretical models for steady flow over moving walls. Stewartson[9] used for the point of zero skin friction the term "separation" of the boundary layer (x_S) and for the point where "the main stream is observed to leave the neighborhood of the wall," the term "breakaway" (x_B). He also noted that the points x_B and x_S may not coincide which is equivalent to the arguments of Refs. 3–5.

The question was and unfortunately remains in many cases embarassing: If the vanishing of the skin friction is not the appropriate criterion for separation, then what should be an acceptable substitute, equally effective, unambiguous and practical? It should be emphasized here that Prandtl's criterion of separation holds for the case of steady flow over fixed walls, for which it was introduced, even within the framework of the full Navier-Stokes equations and has also proved a valuable tool for predicting separation experimentally. A correct definition and a convenient criterion for unsteady separation is therefore needed to fill the void not only for the boundary-layer theory but for the general field of fluid mechanics.

The importance of the preceding work was duly recognized, but for one more decade no progress was made in solving this difficult problem. The interest revived in 1971. Despard and Miller[10] performed some experiments for inviscid flows that oscillate about a mean value and suggested a criterion for the location of separation which was found in this case to be practically independent of time. Stuart,[11] describing the flow about a cylinder started impulsively from rest, refers to the separation bubble at the rear of the cylinder which is still within the boundary layer. He then "hesitates to describe this boundary-layer phenomenon" as "separation," since "it does not involve the large-scale breaking-away of fluid, which we tend to associate with that term." Sears and Telionis[12,13] proposed a definition and a theoretical model for unsteady separation based on the concept of Goldstein's singularity. Buckmaster[14] studied the initial unsteady evolution of the singularity at separation by considering a small impulsive change of the pressure gradient in the vicinity of the separation point. Ruiter et al.[15] have examined with flow visualization techniques the separation of unsteady boundary layers over oscillating air foils. Tennant,[16] Telionis and Werle[17] and Tsahalis and Telionis[18] have later examined the behavior of steady boundary-layer separation with nonvanishing wall conditions.

Finally, Telionis, Tsahalis and Werle[19] have developed a numerical scheme of integration that would proceed through regions of partially reversed flow and considered smooth time-dependent outer flows. They also demonstrated numerically that the point of zero skin friction is nonsingular, but a separation singularity as defined by Sears and Telionis[12,13] develops at a downstream station for pressure gradients that become more adverse as time increases.

In the present paper we employ the methods introduced in Ref. 19 to study the response of the unsteady boundary layer and separation to impulsive changes of the outer flow. Aside from the fact that starting the calculation in this case involves a different technique, these problems are particularly interesting because a thin layer of reversed flow that covers completely the solid boundaries is encountered immediately after the impulsive change of the outer flow. Yet it is shown that the boundary-layer equations can still be integrated marching in the downstream direction.

2. Boundary-Layer Equations

If the freestream velocity U_∞ and a typical length L of the problem are used to nondimensionalize time, velocities and distances and if further the normal coordinate is stretched with the square root of the Reynolds number, the unsteady boundary-layer equations become

$$(\partial u/\partial s)+(\partial v/\partial N) = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial s} + v\frac{\partial u}{\partial N} = \frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial s} + \frac{\partial^2 u}{\partial N^2} \tag{2}$$

The corresponding boundary conditions are

$$u(s,0,t) = v(s,0,t) = 0 \tag{3}$$

$$u(s,\infty,t) \to U_e(s,t) \tag{4}$$

where u, v, and s, N are the velocity components and the coordinates parallel and perpendicular to the wall, respectively, and $U_e(s,t)$ is the outer flow velocity distribution. In this paper, we will consider only impulsive changes of the outer flow from some initial distribution $U_{e1}(s)$ to a final $U_{e2}(s)$. Let us now introduce a Görtler[20] transformation based on the new distribution $U_{e2}(s)$

$$\xi = \int_0^s U_{e2}(s)\,ds \tag{5}$$

$$\eta = [U_{e2}(s)/(2\xi)^{1/2}]N \tag{6}$$

We further replace the dependent variables by

$$F = u/U_e \tag{7}$$

$$V = [(2\xi)^{1/2}/U_e][v+(\partial\eta/\partial s)(2\xi)^{1/2}F] \tag{8}$$

and finally transform the continuity and momentum equations into

$$2\xi(\partial F/\partial\xi)+F+(\partial V/\partial\eta) = 0 \tag{9}$$

$$\frac{2\xi}{U_e^2}\frac{\partial F}{\partial t} + 2\xi F\frac{\partial F}{\partial\xi} + V\frac{\partial F}{\partial\eta} + \beta(F^2-1) = \frac{\partial^2 F}{\partial\eta^2} \tag{10}$$

where β is the pressure gradient function defined by

$$\beta = (2\xi/U_e)(dU_e/d\xi) \tag{11}$$

The initial conditions in terms of the new variables are

$$F(\xi,\eta,0) = F_H(\xi,\eta) \tag{12}$$

$$V(\xi,\eta,0) = V_H(\xi,\eta) \tag{13}$$

and the boundary conditions are

$$F(\xi,0,t) = V(\xi,0,t) = 0 \tag{14}$$

$$F(\xi,\infty,t) \to 1 \tag{15}$$

3. Impulsive Changes of the Outer Flow

It is well known that a flow generated impulsively at time $t=0$ from rest is at first inviscid, while vortex sheets are generated along all solid boundaries.[21] Vorticity then diffuses away from the boundaries and the flowfield tends asymptotically to a steady-state viscous flow. Watson[22] and Askovic[23] have proved that the process of an impulsive change of the flow is similar to the impulsive start and that the inviscid flow generated by the impulsive change is superimposed on the already existing viscous flow. Again vortex sheets are generated on the solid boundaries that subsequently diffuse into the flow.

Let us apply this theorem to a particularly simple outer flow distribution, the linear distribution considered by Howarth[24]

$$U_e = 1 - As \tag{16}$$

The impulsive change can be implemented by assuming that at time $t=0$ the value of A jumps from A_1 to A_2. Hence

$$U_{e1} = 1 - A_1 s \tag{17}$$

$$U_{e2} = 1 - A_2 s \tag{18}$$

According to the theorem now the change $\Delta U_e = (A_2 - A_1)s$ is inviscid and should be superimposed on the established viscous solution.

If $u_H(s,N)$ and $v_H(s,N)$ correspond to the steady-state solution with an outer flow given by Eq. (17) then at $t=0^+$ the flowfield should be given by

$$u(s,N,0^+) = u_H(s,N) - \Delta U_e(\tag{19}$$

$$v(s,N,0^+) = v_H(s,N) \tag{20}$$

and therefore a vortex sheet is generated on the wall.

Figure 1 depicts schematically this situation. The velocity field given by the Eqs. (19) and (20) will be considered as the initial flowfield for the unsteady problem. One of the advantages of this particular outer flow distribution is that for $t \leqq 0$ and $t \to \infty$ the flowfield corresponds to two solutions of the family considered by Howarth. These are steady-state solutions for outer flow distributions given by Eqs. (17) and (18), respectively, and can therefore be checked with Ref. 24 or calculated with a steady-

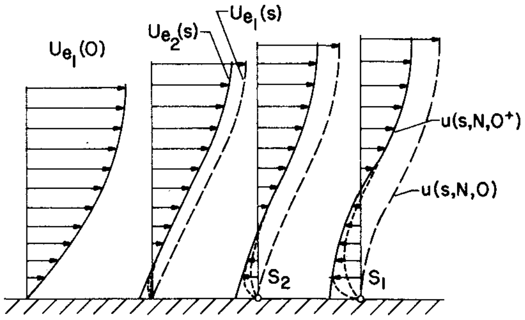

Fig. 1 Schematic of the impulsive change of the flow. The dotted lines denote the expected change after $t = 0^+$ due to vorticity diffusion.

state scheme of integration. For these two extreme cases now the location of separation is unambiguously defined as the point of zero skin friction. Howarth's calculations have shown that the distance to separation is given by

$$S_{zf} \simeq 0.120/A \tag{21}$$

The purpose of this paper is to study the motion of the point of separation as it departs from the steady-state location S_1 (see Fig. 1) corresponding to $A = A_1$ and asymptotically tends to the steady-state location S_2 corresponding to $A = A_2$.

The numerical analysis encounters a difficulty for small times and very close to the wall. This difficulty is due to the inability of the numerical scheme to represent a vortex sheet. An approximate analytical solution for small times was therefore derived and incorporated in the computer program to help start the calculations.

It is known that for impulsive changes and for very small times the convection terms of the momentum equation are much smaller than the unsteady and viscous terms. We can therefore approximate the problem in the nondimensional form with

$$\partial u/\partial t = \partial^2 u/\partial N^2 \tag{22}$$

$$u(s, N, 0) = u_H(s, N) - (U_{e_1} - U_{e_2}) \tag{23}$$

$$u(s, 0, t) = 0 \tag{24}$$

$$u(s, N, t) \to U_{e2}(s) \quad \text{as} \quad N \to \infty \tag{25}$$

where u_H is the solution of the steady-state problem corresponding to a Howarth outer flow distribution given by Eq. (17). The solution to the preceding problem is

$$u(s, N, t) = u_H(s, N) - (A_2 - A_1)s \operatorname{erf}[N/2(t)^{1/2}] \tag{26}$$

The steady-state equations with U_{e1} given by Eq. (17) were first integrated numerically in order to store the quantities $u_H(s, N)$ and $v_H(s, N)$. Then $u_H(s, N)$ was replaced by Eq. (26) evaluated at a very small t, U_{e1} was replaced by U_{e2} with $A_2 > A_1$ and the unsteady program was left to continue the numerical integration.

One of the most interesting features of the time-dependent flowfield under consideration is that at $t = 0^+$ the point of zero skin friction "jumps" from S_1 to $s = 0$. As time tends to 0^+ from above, of course, we recover the vortex sheet which is equivalent to a wall shear tending to $-\infty$. We will now prove that the first point where the shear will increase to zero is indeed the point $s = 0$. The wall shear by virtue of Eq. (26) is given by

$$\left.\frac{\partial u}{\partial N}\right|_{N=0} = \left.\frac{\partial u_H}{\partial N}\right|_{N=0} - (A_2 - A_1)\frac{s}{2(t)^{1/2}} \tag{27}$$

We therefore observe that for a fixed station s, the wall shear can be made as small as we please for small enough time t. That is

$$\left.\frac{\partial u}{\partial N}\right|_{N=0} \to -\infty \quad \text{as} \quad t \to 0^+ \tag{28}$$

It remains to compare two values of the wall shear for fixed time t_0 and two stations s_I and s_{II} where $s_I < s_{II}$.

The difference of the quantities

$$\frac{\partial u(s_I, 0, t_0)}{\partial N} - \frac{\partial u(s_{II}, 0, t_0)}{\partial N} \tag{29}$$

can be written according to Eq. (27) as

$$\left.\frac{\partial u_H}{\partial N}\right|_{s_I} - \left.\frac{\partial u_H}{\partial N}\right|_{s_{II}} - \frac{k}{2(t)^{1/2}}(s_I - s_{II}) \tag{30}$$

But the solution of the problem for u_H has indicated that the shear $\partial u_H/\partial N|_{N=0}$ varies with a power r of s less than unity. This implies that the Quantity (30) becomes positive as $s_I - s_{II}$ tends to zero.

We, therefore, conclude that for $N = 0$, and $t \to 0^+$ the wall shear is everywhere negative and absolutely smaller at s_I than at s_{II}, if $s_I < s_{II}$. The zero of the wall shear therefore tends to $s = 0$ as $t \to 0^+$ from above. Any claim that this should be the behavior of separation should definitely disturb the conscience of any fluid dynamicist, because it would imply that the wake suddenly expands and embraces the whole body. We should mention here that this problem and the description of this section were originally conceived by Prof. W. R. Sears who often used this example in order to point out how misleading the criterion of zero skin friction can be in unsteady flows.

4. Numerical Integration

Equation (10) is written as

$$F_{\eta\eta} + \alpha_1' F_\eta + \alpha_2' F + \alpha_3' + \alpha_4' F_\xi + \alpha_5' F_t = 0 \tag{31}$$

where the coefficients α_i' depend on F, V, and β. The time derivative can be written in a difference form as

$$\partial F/\partial t = [F(\xi, \eta, t) - F(\xi, \eta, t - \Delta t)]/\Delta t \tag{32}$$

and with the notation $F^0 = F(s, \eta, t - \Delta t)$, Eq. (31) can be written as

$$F_{\eta\eta} + \alpha_1 F_\eta + \alpha_2 F + \alpha_3 + \alpha_4 F_\xi = 0 \tag{33}$$

where now

$$\alpha_1 = -V \tag{34}$$

$$\alpha_2 = -\beta F - [2\xi/(1 - As)^2] \cdot (1/\Delta t) \tag{35}$$

$$\alpha_3 = \beta + [2\xi/(1 - As)^2] \cdot (F^0/\Delta t) \tag{36}$$

$$\alpha_4 = -2\xi F \tag{37}$$

Equation (33) has now the form of the steady-state equation and was solved numerically by subroutines developed by Werle and Davis.[25] The scheme of calculation used here was capable of integrating through regions of partially reversed flow. This was accomplished by replacing the ξ derivative at the kth station with

$$\left.\frac{\partial F}{\partial \xi}\right|_k = \frac{1}{2\xi}[(F_{k+1}{}^0 - F_k{}^0) + (F_k - F_{k-1})] \tag{38}$$

Expression (38) was found to be the most effective upwind differencing scheme as reported in Ref. 19.

Our calculations were performed in the following order. First the steady-state equation was integrated with an outer flow distribution given by Eq. (17). Then the flowfield

$$u(\xi, \eta, 0^+) = u_H(\xi, \eta) - \Delta U_e(\xi) \tag{39}$$

$$v(\xi, \eta, 0^+) = v_H(\xi, \eta) \tag{40}$$

was calculated. The approximate solution given by Eq. (26) was evaluated at a small value t_ε and this flowfield was stored to provide the initial condition for integration with respect to time. Time was then incremented to $t_\varepsilon + \Delta t$ and the (ξ, η) plane was swept again in the downstream direction using the data stored at $t = t_\varepsilon$ in order to form time and space derivatives according to Eqs. (32) and (38). The new data at time $t = t_\varepsilon + \Delta t$ replaced those at time $t = t_\varepsilon$ in the storage; time was then incremented again to $t = t_\varepsilon + 2\Delta t$ and this process was repeated for all subsequent time steps. Note that in effect this scheme actually calculates the diffusion of vorticity generated impulsively on an already existing nonsimilar viscous flow.

5. Results and Discussion

In the present work we have used as a criterion for separation the appearance of the separation singularity (see Refs. 12 and 13).

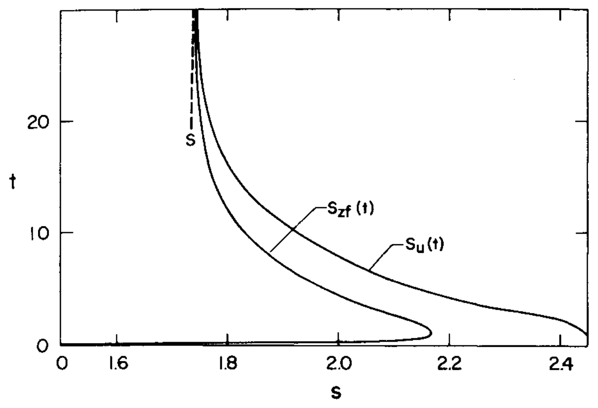

Fig. 2 Temporal path of zero skin friction and separation for an impulsive change with $A_1 = 0.05$ and $A_2 = 0.07$.

Goldstein[26] of course has considered only the case of steady flow and demonstrated that the boundary-layer equations may accept a singular solution at the point of zero skin friction. It was shown though in Refs. 17–19 and Ref. 27 that in unsteady flow the point of zero skin friction is not singular and that all the typical features of a Goldstein singularity appear at the point which Sears and Telionis define as separation. Therefore, it is not just the breakdown of our calculations but rather the appearance of all the familiar properties of the Goldstein singularity that were used to define the location of separation. Typical properties of the separation singularity are for example the growth of the v-component of the velocity or of the displacement thickness, etc., with a fractional power of the distance from separation. The properties of this singularity in steady or unsteady flow were studied in detail in Ref. 19.

The numerical analysis with $A_1 = 0.05$ and $A_2 = 0.070$ has indicated that for small times the vortex sheet governs the whole field and the typical features of the erf function are still present. This can be observed in Fig. 2 where the point of zero skin friction $S_{zf}(t)$ appears to be traveling downstream at a very large rate for $t < 1$ while for this time interval the location of the singularity appears practically stationary. Notice that at $t = 0^+$ the flowfield expressed by the Eqs. (39) and (40) has a point of singularity at the initial point of separation which is contained in the function $u_H(\xi, \eta)$. This singularity is interpreted as unsteady separation $S_u(t)$ and therefore, the analysis indicates that the location of separation and hence the beginning of the wake starts moving very slowly at first towards the upstream direction.

For times larger than 1, $t > 1$, the point S_{zf} reverses its direction and starts traveling upstream. At this instant the location of separation has started moving at a larger rate in the upstream direction too. The two paths for large times are shown to approach asymptotically the location S_2 of steady-state separation corresponding to the new outer flow distribution U_{e2}.

The peculiar behavior of $S_{zf}(t)$ for small times may appear erroneous to the reader and for this reason the same calculation has been repeated with a very small time step and for $t < 1$. The function $S_{zf}(t)$ is plotted again in Fig. 3 with a magnified time scale. Note that the point of zero skin friction appears for a certain time interval to be stationary at about $s = 2.1$, before it starts again moving in the upstream direction. The physical interpretation of this phenomenon is based on the characteristic properties of vortex diffusion. It is known that the accumulated vorticity on the wall affects mostly its neighboring points and therefore the wall shear distribution is governed at first by the process of vorticity diffusion. The negative infinite slopes tend to increase rapidly towards zero and at some points towards positive values. On the other hand the change of the flow due to the new pressure distribution has actually the opposite trend. After all the slopes of the velocity profiles have to decrease well before S_2, so that the zero skin friction criterion for separation is met at $s = S_2$ as $t \to \infty$, in a region where the skin friction was positive at $t = 0$.

To demonstrate more clearly the fact that the two effects of the vorticity diffusion and the pressure gradient are counteracting each other, we have plotted in Figs. 4 and 5 the velocity profiles at $s = 2.000$ and $s = 2.237$, for various instances t. Note that the rate of increase of the velocity with time is smaller for points farther away from the wall. In fact the numerical results indicate that for very large η, the velocity profiles are crossing each other (this is not shown in Figs. 4 and 5).

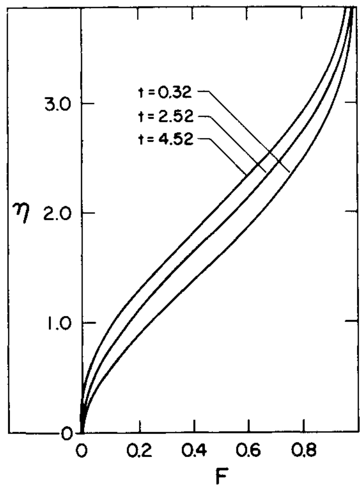

Fig. 4 Velocity profiles at $s = 2.000$ for an impulsive change corresponding to $A_1 = 0.05$ and $A_2 = 0.075$.

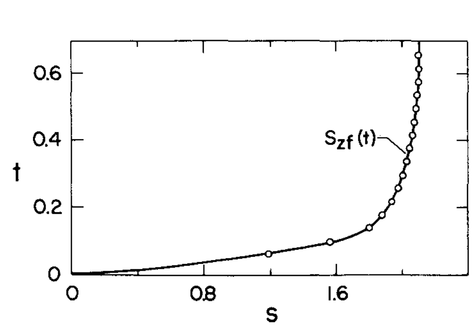

Fig. 3 Detail of the temporal path of zero skin friction for a very small time step of integration $t = 0.04$ and $A_2 = 0.075$.

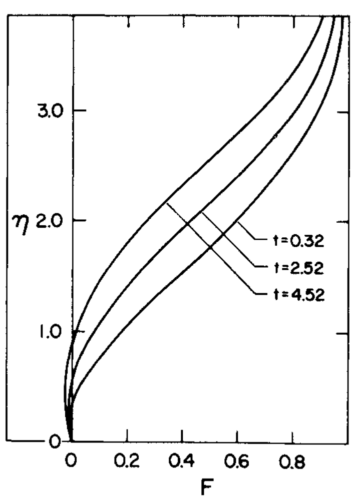

Fig. 5 Velocity profiles at $s = 2.237$ for an impulsive change corresponding to $A_1 = 0.05$ and $A_2 = 0.075$.

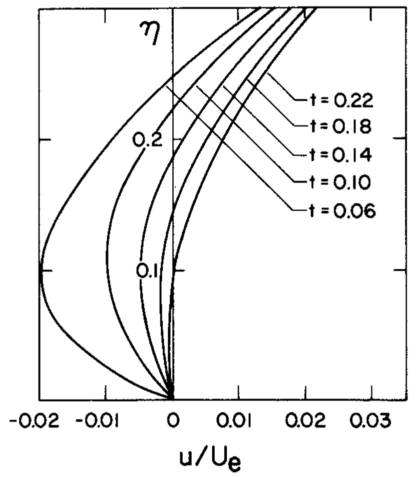

Fig. 6 Detail of velocity profiles at $s = 2.00$ for an impulsive change corresponding to $A_1 = 0.05$ and $A_2 = 0.075$.

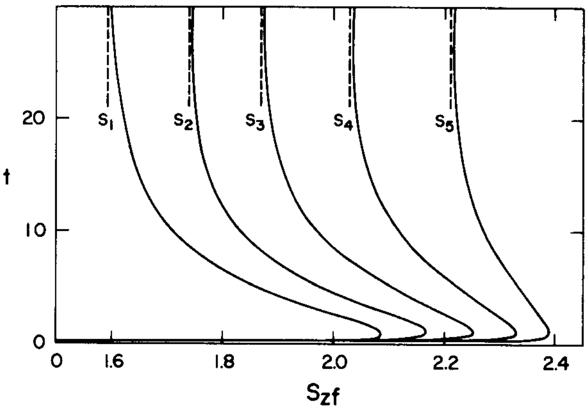

Fig. 8 Temporal paths of zero skin friction for an impulsive change with $A_1 = 0.05$ and $A_2 = 0.075$, 0.070, 0.065, 0.060, and 0.055, corresponding to S_1, S_2, S_3, S_4, and S_5, respectively.

In Figs. 6 and 7 we have plotted a detail of the velocity profiles for a very small η and for very small times in order to demonstrate more clearly the effect of the vorticity diffusion.

In Fig. 8 we have plotted the paths of the zero skin friction in time, $S_{zf}(t)$ for $A_1 = 0.05$ and various values of A_2. In Fig. 9 we have plotted the corresponding paths of unsteady separation $S_u(t)$. In all cases both curves tend asymptotically to the location of steady-state separation corresponding to the outer free distribution given by Eq. (18), that is, the points S_1, S_2, S_3, S_4, and S_5. It should be noted that the point of separation remains practically at its original location S_1 for the initial time interval of $\Delta t \approx 1$. It is evident from the plot of the zero skin friction that this interval corresponds to an inner time region which is dominated by vortex diffusion.

6. Conclusions

The present results have demonstrated that the boundary-layer approximation does not break down, if a portion of the velocity profile is reversed. In fact, in the case of impulsive changes of the outer flow considered here, a thin film of reversed flow may cover the solid boundary of the entire flowfield. It should be noted here that the self similar solutions of the boundary layer include profiles with reversed flow. The particular problem we considered in this paper though involved the appearance of a thin layer of reversed flow in all or part of the flowfield and a method of numerical calculation that marches downstream through such regions. It was also indicated that for small times the location of separation is rather insensitive to changes of the outer flow, at least for $\Delta t \lesssim 1$. Despard and Miller[10] have arrived qualitatively at a similar conclusion. They found that the location of separation is shifted upstream of its steady-state location for an oscillating outer flow but appears to be fixed at this location even though other properties of the flowfield as, for example, the skin friction, follow with some time lag, the oscillations of the freestream.

The fact remains that up to now the present methods have provided perhaps convincing arguments, but no definite proof, that the singularity which was studied theoretically and numerically corresponds to unsteady separation. The only way that this could be verified would be by comparison with numerical results of the full Navier-Stokes equations, or experimental data. The reader is cautioned here to the fact that no experimental method of investigation of the above type could ever verify the properties of the separation singularity which is a characteristic feature of the boundary-layer equation. What we expect to compare is only the location of separation and perhaps some of the typical features of the flow, that signal the fact that separation is approached a little upstream of the actual location of separation. It is believed that one of the main contributions of the present work is to indicate that the secrets of the mechanism of separation, steady or unsteady, are hidden within the thin viscous layer and therefore it is recommended that any experimental investigation should be equipped with the means of magnifying the N-direction of the flowfield.

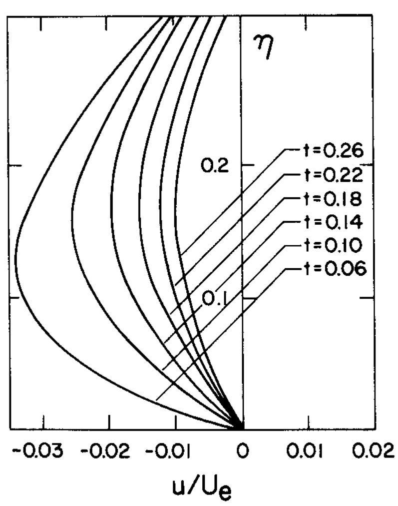

Fig. 7 Detail of velocity profiles at $s = 2.237$ for an impulsive change corresponding to $A_1 = 0.05$ and $A_2 = 0.075$.

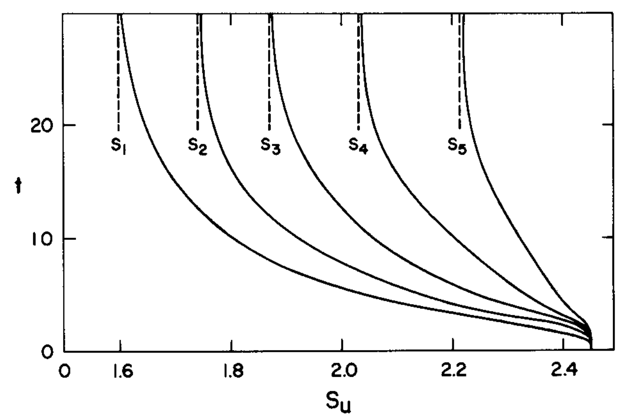

Fig. 9 Temporal paths of separation for an impulsive change with $A_1 = 0.050$ and $A_2 = 0.075, 0.070, 0.065, 0.060$, and 0.055, corresponding to S_1, S_2, S_3, S_4, and S_5, respectively.

Finally, we should mention here that the present method has a disadvantage which is actually inherent in the mathematical model of the unsteady boundary-layer equations. Our scheme of integration forms time derivatives by using the data of the previous time step. Let $S_u(t)$ and $S_u(t+\Delta t)$ be the distances to separation at times t and $t+\Delta t$, respectively. If the location of separation propagates upstream, that is if $S_u(t+\Delta t) < S_u(t)$, then the information stored in the interval $I = [s = S_u(t+\Delta t), s = S_u(t)]$ is not needed for the calculations. To the contrary, if the point of separation moves downstream, that is if $S_u(t+\Delta t) > S_u(t)$ then the calculations require information in the interval I at time t, which now corresponds to the wake and thus is not available. Therefore, the present method is confined only to cases of upstream motions of separation.

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